On certain Possible Cases of Steady Motion in a Viscous Fluid.

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THE following paper contains, first, some general principles governing steady motion in viscous fluids; second, the detailed working out of two cases, (*i*) a sphere moving with constant velocity in the direction of the axis of x, (*ii*) an ellipsoid moving uniformly in the same direction. The results obtained are certain to hold for slow motions, though they have been obtained without that assumption, but it is not proved that the prescribed conditions will exist for rapid motions. If it can be shown that a velocity can be chosen for the moving body, so that the quantity

$$\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz$$

shall be an exact differential, then the solution below given will hold for that case, and for that case only.

Part of what immediately follows I have already given in another place, but it is repeated here for convenience.

The expressions for the fluid pressure in different cases are given in the "Journal of the Franklin Institute," for October, 1880. The values found for the velocities of a fluid particle when a sphere moves in any direction in the fluid are given in the "Philosophical Magazine" for November, 1880. A slight error exists in these values as there given, which is corrected here.

Denote by u, v, w the component velocities of a fluid particle in the direction of the axes x, y, z; p, the density of the fluid at the point x, y, z; ρ, the constant density, and μ, the coefficient of viscosity; the kinematic coefficient of viscosity or the ratio of μ to ρ will be denoted by k, i. e. $\frac{\mu}{\rho} = k$.

The equations of motion of an incompressible viscous fluid are now

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{dp}{dx} + k\Delta^2 u,$$
$$\frac{dv}{dt} = Y - \frac{1}{\rho}\frac{dp}{dy} + k\Delta^2 v, \qquad (1)$$
$$\frac{dw}{dt} = Z - \frac{1}{\rho}\frac{dp}{dz} + k\Delta^2 w,$$

X, Y, Z being external forces. In what follows we will suppose the forces X, Y, Z to possess a potential. If the motion of the fluid is caused by a body which has been projected in it, and is acted upon by forces due to a potential, the potential must be of the form

$$Ax + By + Cz,$$

for the motion to be steady relatively to the body, as a constant resistance has then to be overcome. If the body is at rest and the liquid streaming past it, the potential must contain a term of the form

$$Ax + By + Cz$$

at infinity, to keep up the steady motion; otherwise the motion would die away and the liquid come to rest from the presence of factors of the form e^{-pt}.*

Denoting by ξ, η, ζ the component angular velocities of the fluid particle at the point x, y, z, we have

$$\xi = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),$$
$$\eta = \frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \qquad (2)$$
$$\zeta = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right).$$

From these follow readily the known relations

$$\Delta^2 u = 2\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right),$$
$$\Delta^2 v = 2\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right), \qquad (3)$$
$$\Delta^2 w = 2\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right).$$

* I am indebted to Mr. Greenhill of Emanuel College, Cambridge, for the above remarks, and also for many other most valuable suggestions.

Denote by Ω the resultant angular velocity; then

$$\Omega^2 = \xi^2 + \eta^2 + \zeta^2; \tag{4}$$

also write

$$2\,q = u^2 + v^2 + w^2. \tag{5}$$

The internal friction involves a certain dissipation of energy; the function expressing the rate of dissipation per unit volume has been called by Lord Rayleigh the "dissipation-function;" denoting this by E, we have

$$\begin{aligned} E = 2\,\mu \Bigg\{ \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + \frac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right)^2 \\ + \frac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right)^2 \Bigg\}. \end{aligned} \tag{6}$$

This can be given in a different form by obtaining the expression for $\Delta^2 q$; this is readily found to be

$$\begin{aligned} \Delta^2 q = {} & u\Delta^2 u + v\Delta^2 v + w\Delta^2 w \\ & + \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \\ & + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dv}{dz}\right)^2 \\ & + \left(\frac{dw}{dx}\right)^2 + \left(\frac{dw}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2. \end{aligned} \tag{7}$$

Add $2\,\Omega^2$ to $\frac{E}{2\,\mu}$, and we will eliminate terms of the form

$$\frac{dw}{dy} \cdot \frac{dv}{dz}, \text{ etc.}$$

Then compare the resulting form of equation (6) with equation (7), and we have at once

$$E = 2\,\mu\{\Delta^2 q - (u\Delta^2 u + v\Delta^2 v + w\Delta^2 w) - 2\,\Omega^2\} \tag{8}$$

or

$$u\Delta^2 u + v\Delta^2 v + w\Delta^2 w = \Delta^2 q - \frac{1}{2\,\mu}E - 2\,\Omega^2. \tag{9}$$

In equations (1) the quantities on the left-hand sides may be replaced by

$$\frac{du}{dt} + u\,\frac{du}{dx} + v\,\frac{dv}{dx} + w\,\frac{dw}{dx} + 2\,(w\eta - v\zeta), \text{ etc.,}$$

or by

$$\frac{du}{dt} + \frac{dq}{dx} + 2\,(w\eta - v\zeta), \text{ etc.}$$

On making these changes, the equations of motion become

$$\begin{aligned}
&\frac{du}{dt} + \frac{d(V+q)}{dx} + \frac{1}{\rho}\frac{dp}{dx} + 2(w\eta - v\zeta) = k\Delta^2 u,\\
&\frac{dv}{dt} + \frac{d(V+q)}{dy} + \frac{1}{\rho}\frac{dp}{dy} + 2(u\zeta - w\xi) = k\Delta^2 v,\\
&\frac{dw}{dt} + \frac{d(V+q)}{dz} + \frac{1}{\rho}\frac{dp}{dz} + 2(v\xi - u\eta) = k\Delta^2 w.
\end{aligned} \tag{10}$$

To these is to be added the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{11}$$

Write

$$P = V + q + \int \frac{dp}{\rho};$$

then, introducing the conditions for steady motion, (10) become

$$\begin{aligned}
&\frac{dP}{dx} + 2(w\eta - v\zeta) = k\Delta^2 u,\\
&\frac{dP}{dy} + 2(u\zeta - w\xi) = k\Delta^2 v,\\
&\frac{dP}{dz} + 2(v\xi - u\eta) = k\Delta^2 w.
\end{aligned} \tag{12}$$

If we assume that the quantities $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$ are the first differential coefficients with respect to x, y, z of a function Q, these equations become

$$\begin{aligned}
&\frac{d(P-kQ)}{dx} = -2(w\eta - v\zeta),\\
&\frac{d(P-kQ)}{dy} = -2(u\zeta - w\xi),\\
&\frac{d(P-kQ)}{dz} = -2(v\xi - u\eta).
\end{aligned} \tag{13}$$

Multiplying these by u, v, w respectively, and then by ξ, η, ζ, and in each case adding the results, we have, writing for brevity

$$\Theta = P - kQ,$$

$$\begin{aligned}
&u\frac{d\Theta}{dx} + v\frac{d\Theta}{dy} + w\frac{d\Theta}{dz} = 0,\\
&\xi\frac{d\Theta}{dx} + \eta\frac{d\Theta}{dy} + \zeta\frac{d\Theta}{dz} = 0,
\end{aligned} \tag{14}$$

and also

$$\frac{d\Theta}{dn} = q'\Omega \sin \delta, \tag{15}$$

where $q' = \sqrt{2} \, q$ is the current velocity, and δ is the angle between the stream line and the vortex line at the point x, y, z. Hence the conditions that the state of motion of the fluid for which

$$\Delta^2 u \, . \, dx + \Delta^2 v \, . \, dy + \Delta^2 w \, . \, dz$$

is an exact differential, are as follows: It must be possible to draw in the fluid a system of surfaces, $\Theta =$ const., infinite in number, and each of which is covered by a network of stream lines and vortex lines. This is the property denoted by equations (14). The product $q'\Omega \sin \delta \, dn$ must be constant over each such surface, dn denoting the length of the normal drawn to the consecutive surface of the system. These results are identical in form with those given for a perfect fluid by Professor Lamb in his work on Fluid Motion. In order that

$$\Delta^2 u \, . \, dx + \Delta^2 v \, . \, dy + \Delta^2 w \, . \, dz$$

shall be an exact differential, the equations of condition

$$\Delta^2 \xi = 0, \qquad \Delta^2 \eta = 0, \qquad \Delta^2 \zeta = 0, \tag{16}$$

must hold.

If we assume that the motion of the fluid is so slow that squares and products of the velocities may be neglected, equations (1) become, when a potential exists,

$$\left.\begin{aligned} k\Delta^2 u &= \frac{dU}{dx}, \\ k\Delta^2 v &= \frac{dU}{dy}, \\ k\Delta^2 w &= \frac{dU}{dz}, \end{aligned}\right\} \tag{17}$$

where

$$U = V + \int \frac{dp}{\rho}. \tag{18}$$

In this case the quantity

$$\Delta^2 u \, . \, dx + \Delta^2 v \, . \, dy + \Delta^2 w \, . \, dz$$

is obviously an exact differential, and the equations of condition

$$\Delta^2 \xi = 0, \qquad \Delta^2 \eta = 0, \qquad \Delta^2 \zeta = 0,$$

are satisfied.

From equations (12) we have, in every case,

$$u\left(\frac{dP}{dx}-k\Delta^2u\right)+v\left(\frac{dP}{dy}-k\Delta^2v\right)+w\left(\frac{dP}{dz}-k\Delta^2w\right)=0,$$
$$\xi\left(\frac{dP}{dx}-k\Delta^2u\right)+\eta\left(\frac{dP}{dy}-k\Delta^2v\right)+\zeta\left(\frac{dP}{dz}-k\Delta^2w\right)=0, \tag{19}$$

so that the conditions for steady motion hold now as in the particular case just mentioned; but in this case the surfaces in the fluid are given by the differential equation

$$dP-k\,(\Delta^2u\,.\,dx+\Delta^2v\,.\,dy+\Delta^2w\,.\,dz)=0. \tag{20}$$

Write for convenience

$$\begin{aligned} L &= v\zeta-w\eta,\\ M &= w\xi-u\zeta,\\ N &= u\eta-v\xi. \end{aligned} \tag{21}$$

Equations (12) now become

$$\begin{aligned} \frac{dP}{dx}-2\,L &= k\Delta^2u,\\ \frac{dP}{dy}-2\,M &= k\Delta^2v,\\ \frac{dP}{dz}-2\,N &= k\Delta^2w; \end{aligned} \tag{22}$$

and from these, by differentiating for x, y, z, respectively, and adding, we have

$$\Delta^2P=2\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right). \tag{23}$$

The same equation holds when the motion is not steady; for if we differentiate equations (10) for x, y, z, respectively, and add, the terms containing the differential coefficients of u, v, w with respect to t will disappear by virtue of the equation of continuity. Integrating (23), and substituting for P its value, there results

$$\frac{p}{\rho}=G-\left\{\frac{1}{2\pi}\iiint\left(\frac{dL}{dx'}+\frac{dM}{dy'}+\frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r}+V+q\right\}. \tag{24}$$

In the general case G is a function of the time; for steady motion, however, it is a constant. The quantities L, M, N may be the first differential coefficients with respect to x, y, z of function of x, y, z; suppose such a function Ψ to exist that we have

$$\begin{aligned} 2\,L &= \frac{d\Psi}{dx},\\ 2\,M &= \frac{d\Psi}{dy},\\ 2\,N &= \frac{d\Psi}{dz}, \end{aligned} \tag{25}$$

equations (22) become in this case

$$\left.\begin{aligned}\frac{d(P-\Psi)}{dx} &= k\Delta^2 u,\\ \frac{d(P-\Psi)}{dy} &= k\Delta^2 v,\\ \frac{d(P-\Psi)}{dz} &= k\Delta^2 w\,;\end{aligned}\right\} \tag{26}$$

and from these results

$$\Delta^2(P-\Psi)=0, \tag{27}$$

and consequently

$$\frac{p}{\rho}=G-\left\{\frac{1}{4\pi}\iiint\frac{\Delta_1^2\Psi}{r}\,dx'dy'dz'+V+q\right\}, \tag{28}$$

in which

$$\Delta_1^2\equiv\frac{d^2}{dx'^2}+\frac{d^2}{dy'^2}+\frac{d^2}{dz'^2}. \tag{29}$$

Equation (24) can be thrown into another form by very simple transformations. Write

$$D^2=L^2+M^2+N^2; \tag{30}$$

then, denoting by α, β, γ the direction-cosines of the vector D,

$$\left.\begin{aligned}L&=\alpha D,\\ M&=\beta D,\\ N&=\gamma D.\end{aligned}\right\} \tag{31}$$

Substituting in (30) the values of L, M, N, we have

$$D=q'\Omega\left[1-\left(\frac{u\xi}{q'\Omega}+\frac{v\eta}{q'\Omega}+\frac{w\zeta}{q'\Omega}\right)^2\right], \tag{32}$$

or

$$D=q'\Omega\sin\delta, \tag{33}$$

where δ is the angle between the stream line and the vortex line at the point x, y, z. Now let a, b, c denote the direction-cosines of a normal to the closed surface containing the fluid; then

$$a\alpha+b\beta+c\gamma$$

is the sine of the angle between the normal and the plane containing the instantaneous axis of rotation Ω and the direction of the resultant velocity q', or

$$\sin\phi=a\alpha+b\beta+c\gamma. \tag{34}$$

Similarly, if we denote by a', b', c' the direction-cosines of the line joining (x, y, z)

to (x', y', z') and ϕ' the angle between this line and the above-mentioned plane, we have

$$\sin\phi' = a'\alpha + b'\beta + c'\gamma. \tag{35}$$

Take now the triple integral in (24),

$$-\frac{1}{2\pi}\iiint\left(\frac{dL}{dx'} + \frac{dM}{dy'} + \frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r};$$

this is

$$= \frac{1}{2\pi}\iint(aL + bM + cN)\frac{d\sigma}{r} + \frac{1}{2\pi}\iiint\frac{a'L + b'M + c'N}{r^2}dx'dy'dz',$$

where $d\sigma$ is an element of the bounding surface. By virtue of the above equations, we have

$$aL + bM + cN = (a\alpha + b\beta + c\gamma)D = q'\Omega\sin\delta\sin\phi,$$

and similarly

$$a'L + b'M + c'N = q'\Omega\sin\delta\sin\phi';$$

therefore

$$\begin{aligned}-\frac{1}{2\pi}&\iiint\left(\frac{dL}{dx'} + \frac{dM}{dy'} + \frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r}\\ &= \frac{1}{2\pi}\iint\frac{q'\Omega\sin\delta\sin\phi}{r}d\sigma + \frac{1}{2\pi}\iiint\frac{q'\Omega\sin\delta\sin\phi'}{r^2}dx'dy'dz',\end{aligned} \tag{36}$$

and finally (24) becomes

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi}\iint\frac{q'\Omega\sin\delta\sin\phi}{r}d\sigma + \frac{1}{2\pi}\iiint\frac{q'\Omega\sin\delta\sin\phi'}{r^2}dx'dy'dz'. \tag{37}$$

If the motion in the fluid is a screw motion, i. e. if the direction of motion be along the instantaneous axis of rotation, we shall have $\delta = 0$, and consequently

$$\frac{p}{\rho} = G - (V + q). \tag{38}$$

If the plane containing the direction of motion and the instantaneous axis of rotation be always normal to the bounding surface, we shall have $\phi = 0$, and then

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi}\iiint\frac{q'\Omega\sin\delta\sin\phi'}{r^2}dx'dy'dz'. \tag{39}$$

We will pass now to the consideration of one or two particular cases. Assume, first, that u, v, w are given by the equations

$$u = \frac{dW}{dy} - \frac{dV}{dz},$$

$$v = \frac{dU}{dz} - \frac{dW}{dx}, \tag{40}$$

$$w = \frac{dV}{dx} - \frac{dU}{dy}.$$

The functions U, V, W must, as is well known, satisfy the equations of condition

$$\Delta^2 U = -2\xi, \qquad \Delta^2 V = -2\eta, \qquad \Delta^2 W = -2\zeta,$$

$$\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = 0. \tag{41}$$

Instead of the three functions U, V, W, we may introduce a single function Φ and write

$$U = z\frac{d\Phi}{dy} - y\frac{d\Phi}{dz},$$

$$V = x\frac{d\Phi}{dz} - z\frac{d\Phi}{dx}, \tag{42}$$

$$W = y\frac{d\Phi}{dx} - x\frac{d\Phi}{dy};$$

these quantities will satisfy equations (41), and give us for the values of ξ, η, ζ,

$$\xi = \frac{1}{2}\left(z\frac{d\phi}{dy} - y\frac{d\phi}{dz}\right),$$

$$\eta = \frac{1}{2}\left(x\frac{d\phi}{dz} - z\frac{d\phi}{dx}\right), \tag{43}$$

$$\zeta = \frac{1}{2}\left(y\frac{d\phi}{dx} - x\frac{d\phi}{dy}\right),$$

where

$$\phi = -\Delta^2\Phi.$$

The function ϕ must also satisfy the equation

$$\Delta^2\phi = 0, \tag{45}$$

since

$$\Delta^2\xi = \Delta^2\eta = \Delta^2\zeta = 0.$$

For ϕ we can take any homogeneous function of the n^{th} degree satisfying (45) and then will have

$$x\frac{d\phi}{dx} + y\frac{d\phi}{dy} + z\frac{d\phi}{dz} = n\phi, \tag{46}$$

or ϕ represents a solid spherical harmonic of the degree n. For the values of u, v, w we have now

$$
\begin{aligned}
u &= \frac{d}{dx}\left\{\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + x\phi, \\
v &= \frac{d}{dy}\left\{\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + y\phi, \qquad (47) \\
w &= \frac{d}{dz}\left\{\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + z\phi.
\end{aligned}
$$

Reverting to equations (3) for a convenient form of obtaining the values of $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$, we readily find for these quantities the values

$$
\begin{aligned}
\Delta^2 u &= n\frac{d\phi}{dx}, \\
\Delta^2 v &= n\frac{d\phi}{dy}, \qquad (48) \\
\Delta^2 w &= n\frac{d\phi}{dz}.
\end{aligned}
$$

The function Q of equations (13) is now the function $n\phi$, or Q is in this case a solid spherical harmonic of the degree n. Equations (40) involve the assumption that the motion is purely of a rotational character. If we for a moment abstract the friction in the fluid from consideration, the motion, if caused by a solid moving in the fluid, will be irrotational, and therefore subject to a velocity potential, say ψ, satisfying the equation $\Delta^2\psi = 0$. We can then in general write the values of u, v, w in the forms

$$
\begin{aligned}
u &= -\frac{d\psi}{dx} + \frac{dW}{dy} - \frac{dV}{dz}, \\
v &= -\frac{d\psi}{dy} + \frac{dU}{dz} - \frac{dW}{dx}, \qquad (49) \\
w &= -\frac{d\psi}{dz} + \frac{dV}{dx} - \frac{dU}{dy}.
\end{aligned}
$$

Equations (43), giving the values of the rotation components, will be, of course, unaltered by this change in the values of u, v, w, and we shall have, instead of (47),

$$
\begin{aligned}
u &= \frac{d}{dx}\left\{-\psi + \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + x\phi, \\
v &= \frac{d}{dy}\left\{-\psi + \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + y\phi, \qquad (50) \\
w &= \frac{d}{dz}\left\{-\psi + \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + z\phi.
\end{aligned}
$$

From equations (43) we derive at once

$$\xi \frac{d\phi}{dx} + \eta \frac{d\phi}{dy} + \zeta \frac{d\phi}{dz} = 0, \tag{51}$$

from which it follows that the vortex lines lie on the surfaces given by the equation $\phi = \text{const.}$ From equations (14) we have, however, since in this case

$$\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz$$

is an exact differential,

$$\xi \frac{d\Theta}{dx} + \eta \frac{d\Theta}{dy} + \zeta \frac{d\Theta}{dz} = 0,$$

and consequently the vortex lines in the fluid lie at the intersection of the surfaces

$$\left.\begin{aligned} \phi &= \text{const.} \\ \Theta &= \text{const.} \end{aligned}\right\} \tag{52}$$

The surfaces Θ are fixed in the fluid, but the surfaces ϕ may move; their motion, however, will always be in such a manner that the above condition shall be satisfied. Equations (50) can be thrown into a simpler form by the following considerations. Write

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

and

$$\log r = \lambda.$$

Then

$$x \frac{d\Phi}{dx} + y \frac{d\Phi}{dy} + z \frac{d\Phi}{dz} = r \frac{d\Phi}{dr};$$

but

$$\frac{d}{dr} = \frac{1}{r} \frac{d}{d\lambda},$$

and consequently

$$r \frac{d\Phi}{dr} = \frac{d\Phi}{d\lambda}.$$

We have then

$$\left.\begin{aligned} u &= \frac{x}{r^2} \frac{d}{d\lambda}\left(\Phi - \psi + \frac{d\Phi}{d\lambda}\right) + x\phi, \\ v &= \frac{y}{r^2} \frac{d}{d\lambda}\left(-\psi + \Phi + \frac{d\Phi}{d\lambda}\right) + y\phi, \\ w &= \frac{z}{r^2} \frac{d}{d\lambda}\left(-\psi + \Phi + \frac{d\Phi}{d\lambda}\right) + z\phi. \end{aligned}\right\} \tag{53}$$

The solution of the problem when the motion of the fluid is caused by a sphere moving through it is quite simple.* We have first to determine the velocity

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* See an article on this subject by the author in the "Philosophical Magazine" for November, 1880.

potential. In Lamb's "Treatise on Fluid Motion" he gives a solution, due to Stokes, of the problem of a sphere moving with uniform velocity in a viscous fluid in the case when the motion of the solid is along the axis of x, and the motion of the fluid is symmetrical around this axis. Special polar co-ordinates are employed in obtaining the required solution, but from the general values above given for u, v, w, we can readily obtain the same results in a very simple manner. We will consider this case for a moment, as the forms of u, v, w thus obtained are of use in another and rather more difficult problem. At an infinitely great distance from the origin the fluid is streaming along the axis of x with a velocity $= -\lambda$, or

$$u = -\lambda,$$
$$v = 0,$$
$$w = 0,$$

for all of the motion which is due to a velocity potential. We have then, as indeed we know from other considerations,

$$\phi = -\lambda x.$$

Our former value of ϕ was

$$\phi = \sum_0^\infty L_i \phi_i,$$

where

$$L_i = 1 + \frac{i}{i+1}\left(\frac{a}{r}\right)^{2i+1};$$

this will now reduce to

$$\phi = \phi_1 = -\lambda x.$$

For the general value of u we take into account the friction terms. The quantities L_i and R_i will all disappear with the exception of L_1 and R_1, and these also vanish at infinity. For these we have the values

$$L_1 = -1 + \frac{1}{2}\left(\frac{a}{r}\right)^3,$$

$$R_1 = -\frac{3}{4}\left(\frac{a}{r}\right)^3 + \frac{3}{4}\frac{a}{r}.$$

For u we have, then,

$$u = \frac{d\phi_1}{dx}\left[L_1 + 2R_1 + r\frac{dR_1}{dr}\right] + \phi_1 \frac{d}{dx}(L_1 - R_1),$$

which, on substitution of the values of L_1, R_1, ϕ_1, becomes

$$u = \lambda\left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right] - \frac{3\lambda}{4}\left[\frac{a}{r^3} - \frac{a^3}{r^5}\right]x^2;$$

or

$$u = \lambda\left[1 - \frac{3}{4}\frac{a}{r}\right] - \frac{3\,\lambda a}{4}\frac{x^2}{r^3} - \frac{\lambda a^3}{4}\left[\frac{r^2 - 3\,x^2}{r^5}\right].$$

But

$$\frac{r^2 - 3\,x^2}{r^5} = \frac{d^2}{dx^2}\cdot\frac{1}{r},$$

and

$$\frac{x^2}{r^3} = x\,\frac{d}{dx}\,\frac{1}{r};$$

Writing then

$$\frac{3\,\lambda a}{4}\cdot\frac{1}{r} = \chi_1,$$

$$\frac{\lambda a^3}{4}\cdot\frac{1}{r} = \chi_2,$$

we have finally

$$u = \lambda - \chi_1 + x\,\frac{d\chi_1}{dx} + \frac{d^2\chi_2}{dx^2},$$

or

$$u = (\lambda - 2\,\chi_1) + \frac{d}{dx}\left(x\chi_1 + \frac{d\chi_2}{dx}\right).$$

Similarly,

$$v = \frac{d}{dy}\left(x\chi_1 + \frac{d\chi_2}{dx}\right),$$

$$w = \frac{d}{dz}\left(x\chi_1 + \frac{d\chi_2}{dx}\right).$$

The determination of the resistance experienced by the sphere, supposing it to move along x with velocity $= +\lambda$, is the same thing as the determination of the pressure upon the sphere supposed at rest and the fluid streaming past it with velocity $= -\lambda$. The latter case is the one that we are dealing with, and to obtain this pressure we use the dissipation-function, employing the form

$$E = 2\,\mu\,\{\Delta^2 q - (u\Delta^2 u + v\Delta^2 v + w\Delta^2 w) - 2\,\Omega^2\}.$$

We had

$$2\,q = u^2 + v^2 + w^2,$$

then

$$2\,\Delta^2 q = \Delta^2 (u^2 + v^2 + w^2);$$

substituting the values of u, v, w, gives

$$2\,q = (\lambda - 2\,\chi_1)^2 + 2\,(\lambda - 2\,\chi_1)\,\frac{dQ}{dx} + \left(\frac{dQ}{dx}\right)^2 + \left(\frac{dQ}{dy}\right)^2 + \left(\frac{dQ}{dz}\right)^2,$$

where for brevity we have written

$$Q = x\chi + \frac{d\chi_2}{dx};$$

to this add

$$a = \frac{3\lambda a}{4},$$

$$\rho = \frac{\lambda a^3}{4};$$

now, introducing the values of Q and X_1, we find readily

$$2\Delta^2 q = a^2 \frac{8r^2 - 12x^2}{r^6} + \beta^2 \frac{36r^2 + 72x^2}{r^{10}} + a\beta \frac{4r^2 - 60x^2}{r^8} + 4a\lambda \frac{3x^2 - r^2}{r^5}.$$

We also find easily

$$2(u\Delta^2 u + v\Delta^2 v + w\Delta^2 w) = a^2 \frac{4r^2 - 20x^2}{r^6} + a\beta \frac{12x^2 + 4r^2}{r^8} + 4a\lambda \frac{3x^2 - r^2}{r^5}$$

and

$$4\Omega^2 = 4a^2 \frac{r^2 - x^2}{r^6}.$$

Combining all of these, we obtain

$$\begin{aligned} E &= \mu \left\{ a^2 \frac{12x^2}{r^6} + \beta^2 \frac{36r^2 + 72x^2}{r^{10}} - 2a\beta \frac{36x^2}{r^8} \right\} \\ &= 12\mu \left\{ \frac{a^2}{r^4} + \frac{9\beta^2}{r^8} - \frac{6a\beta}{r^6} \right\} \frac{x^2}{r^2} - \mu \frac{36\beta^2 x^2}{r^{10}} + \mu \frac{36\beta^2 r^2}{r^{10}} \\ &= 12\mu \left\{ \frac{a}{r^2} - \frac{3\beta}{r^4} \right\}^2 \frac{x^2}{r^2} + \mu \frac{36\beta^2}{r^8} \left\{ 1 - \frac{x^2}{r^2} \right\}. \end{aligned}$$

Writing

$$\frac{x}{r} = \cos\theta,$$

multiplying E by $2\pi r^2 \sin\theta\, d\theta dr$, and integrating from $\theta = 0$ to $\theta = \pi$ and from $r = a$ to $r = \infty$, we have, for the total rate of dissipation of energy,

$$16\pi\mu \left\{ \frac{3\beta^2}{a^5} - \frac{2a\beta}{a^3} + \frac{a^2}{a} \right\} = 6\pi\mu a\lambda^2.$$

If X denote the force which must act upon the sphere in order to keep it at rest, we have

$$X\lambda = 6\pi\mu a\lambda^2,$$

or

$$X = 6\pi\mu a\lambda.$$

These are the results given by Lamb in his treatise. In the general case the velocity u can be written in the form

$$u = \sum_0^\infty \left[\frac{d\phi_i}{dx} \left\{ (2i+1) R_i + r\frac{dR_i}{dr} \right\} - \frac{d}{dx}(L_i\phi_i) - i\frac{d}{dx}(R_i\phi_i) \right]$$

with similar expressions for v and w. Applying the operator Δ^2 to this, we have, if we assume

$$u = u_1 + u_2 + \ldots + u_i,$$

$$\Delta^2 u_i = i \frac{ds_i}{dx},$$

and

$$\Delta^2 u = \frac{d}{dx} \Sigma s_i;$$

also

$$\Delta^2 v = \frac{d}{dy} \Sigma s_i,$$

$$\Delta^2 w = \frac{d}{dz} \Sigma s_i.$$

The other terms vanish on applying this operator, as of course they should do.

In the case where the axis of spin and the direction of the current velocity lie always in a plane normal to the surface of the solid, we have for the determination of the pressure

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi} \iiint \frac{q'\Omega \sin \delta \sin \phi'}{r^2} \, dx' dy' dz'.$$

In the problem just discussed of the motion of the fluid all parallel to the axis of x, we have

$$u\xi + v\eta + w\zeta = 0,$$

or the stream lines and vortex lines are at right angles to each other; this gives

$$\sin \delta = 1,$$

and obviously in this case the plane above mentioned is normal to the surface of the sphere. We have then

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi} \iiint \frac{q'\Omega \sin \phi'}{r^2} \, dx' dy' dz'.$$

Concerning V we have

$$-\frac{dV}{dx} = X,$$

therefore

$$V = 6\,\pi\mu a\lambda x + \text{const.}$$

In the general case, where there is no restriction as to the direction of motion of the fluid, the pressure must be determined by means of the equation

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi}\iint \frac{q'\Omega \sin\delta \sin\phi}{r}\, d\sigma + \frac{1}{2\pi}\iiint \frac{q'\Omega \sin\delta \sin\phi'}{r^2}\, dx'dy'dz'.$$

The computation would certainly be very difficult, if not impossible, from the complicated nature of the quantities involved. The first step, however, would be the determination of V; the angles δ and ϕ can be found from the expressions for the velocities. If the motion is very slow and no external forces act on the fluid, we shall always have

$$\frac{p}{\rho} = \int (\Delta^2 u\,.\,dx + \Delta^2 v\,.\,dy + \Delta^2 w\,.\,dz).$$

Kirchhoff (*vide* Mathematische Physik, p. 377) has solved the problem of an ellipsoid of revolution rotating with constant velocity about its axis in a viscous fluid, both for the cases of an infinite extent of fluid and for a mass of fluid contained within a confocal ellipsoid. I do not see how to attack the general problem of the motion of any ellipsoid in a mass of viscous fluid; but for the case of simple translation along one of the axes it is not difficult to find values for u, v, w which will satisfy all the prescribed conditions. Suppose an ellipsoid in the fluid with its axes coinciding with those of the co-ordinates,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If we assume first the case of no friction, and call V the potential of the ellipsoid at an external point, we have

$$V = \pi abc \int_\sigma^\infty \frac{1 - \dfrac{x^2}{a^2+\psi} - \dfrac{y^2}{b^2+\psi} - \dfrac{z^2}{c^2+\psi}}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}\, d\psi,$$

or

$$V = \text{const.} - 2\pi(Ax^2 + By^2 + Cz^2),$$

where σ is the greatest root of the equation

$$\frac{x^2}{a^2+\sigma} + \frac{y^2}{b^2+\sigma} + \frac{z^2}{c^2+\sigma} = 1.$$

Now for the velocity potential ϕ we have (American Journal of Mathematics, Vol. II. p. 260, et seq.),

$$\phi = -\lambda\left(x - \frac{1}{2\pi(2-A)}\frac{dV}{dx}\right).$$

The quantities A, B, C are known to be

$$A = abc\int_0^\infty \frac{d\psi}{(a^2+\psi)\,N},$$

$$B = abc\int_0^\infty \frac{d\psi}{(b^2+\psi)\,N},$$

$$C = abc\int_0^\infty \frac{d\psi}{(c^2+\psi)\,N},$$

in which

$$N = \sqrt{(a^2+\psi)\,(b^2+\psi)\,(c^2+\psi)}.$$

The velocities u, v, w, in this case, will have for values

$$u = -\lambda + \frac{1}{2\pi(2-A)}\,\frac{d^2V}{dx^2},$$

$$v = \frac{1}{2\pi(2-A)}\,\frac{d^2V}{dxdy},$$

$$w = \frac{1}{2\pi(2-A)}\,\frac{d^2V}{dxdz}.$$

Reverting now for a moment to the case of the sphere, we had

$$u = \lambda - 2\chi_1 + \frac{d}{dx}(x\chi_1) + \frac{d^2\chi_2}{dx^2},$$

$$v = \frac{d}{dy}(x\chi_1) + \frac{d^2\chi_2}{dxdy},$$

$$w = \frac{d}{dz}(x\chi_1) + \frac{d^2\chi_2}{dxdz},$$

in which

$$\chi_2 = \frac{\lambda a^3}{4}\cdot\frac{1}{r},$$

a quantity proportional to the potential of the solid homogeneous sphere upon an external point. The same remark, of course, holds concerning $\frac{V}{2\pi(2-A)}$. Write now for the velocities in the case of the ellipsoid the following system of values similar to those obtained for the sphere:—

$$u = \lambda - 2\Psi + \frac{d}{dx}(x\Psi) + \frac{1}{2\pi(2-A)}\,\frac{d^2V}{dx^2},$$

$$v = \frac{d}{dy}(x\Psi) + \frac{1}{2\pi(2-A)}\,\frac{d^2V}{dxdy},$$

$$w = \frac{d}{dz}(x\Psi) + \frac{1}{2\pi(2-A)}\,\frac{d^2V}{dxdz}.$$

These must satisfy the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

this gives

$$-2\frac{d\Psi}{dx} + \Delta^2(x\Psi) = 0,$$

or simply

$$\Delta^2\Psi = 0.$$

This is satisfied (Ferrer's Spherical Harmonics, p. 110) by assuming

$$\Psi = \int_\sigma^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}},$$

ψ and σ having the meaning already assigned them. For greater convenience we will write

$$V = \text{const.} - (A_1x^2 + B_1y^2 + C_1z^2),$$

where

$$A_1 = 2\pi A, \text{ etc.},$$

also

$$\Psi = 2\pi abc\int_\sigma^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}.$$

This (Ferrers, p. 111) is the potential of a homogeneous ellipsoidal shell of determinate density at an external point. For the density we have (Kirchhoff, p. 179),

$$\frac{d\Psi}{dn_i} + \frac{d\Psi}{dn_\sigma} = -4\pi h,$$

which gives at once

$$h = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1};$$

that is, the density of the ellipsoidal shell at any point is proportional to the central perpendicular upon the tangent plane to the surface at that point. The values given now for u, v, w satisfy all required conditions, and we have in this case also

$$\Delta^2u\,.\,dx + \Delta^2v\,.\,dy + \Delta^2w\,.\,dz,$$

an exact differential, viz. $d\,.\,\Delta^2(x\Psi)$. For greater generality, however, we may introduce two arbitrary constants, say α and β; then

$$u = \lambda - 2\alpha\Psi + \alpha\frac{d}{dx}(x\Psi) + \frac{\beta}{(4\pi - A_1)}\frac{d^2V}{dx^2},$$

$$v = \qquad \alpha\frac{d}{dy}(x\Psi) + \frac{\beta}{(4\pi - A_1)}\frac{d^2V}{dxdy},$$

$$w = \qquad \alpha\frac{d}{dz}(x\Psi) + \frac{\beta}{(4\pi - A_1)}\frac{d^2V}{dxdz}.$$

Now at the surface of the body we have

$$u = v = w = 0,$$

and also at the surface $\sigma = 0$; therefore

$$\Psi_0 = 2\pi abc \int_0^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}.$$

It will be convenient here to make a little digression and give the values of certain of our quantities as elliptic functions. Take ψ_1, ψ_2, ψ_3 as the variable parameters of a system of surfaces confocal to the given ellipsoid; $\delta_1, \delta_2, \delta_3$ as the amplitudes of three elliptic integrals

$$\theta_1 = \int \frac{d\delta_1}{\Delta(k_1\delta_1)}, \qquad \theta_2 = \int \frac{d\delta_2}{\Delta(k_1\delta_2)}, \qquad \theta_3 = \int \frac{d\delta_3}{\Delta(k_1\delta_3)}.$$

We have now

$$x^2 = \frac{(a^2+\psi_1)(a^2+\psi_2)(a^2+\psi_3)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2 = \frac{(b^2+\psi_1)(b^2+\psi_2)(b^2+\psi_3)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2 = \frac{(c^2+\psi_1)(c^2+\psi_2)(c^2+\psi_3)}{(c^2-a^2)(c^2-b^2)}.$$

Write now

$$\psi_1 = c^2 \frac{\frac{a^2-c^2}{c^2} - t^2}{x^2},$$

then make

$$t = \tan \delta_1,$$

where δ_1 lies between 0 and $\frac{\pi}{2}$. Similar transformations for ψ_2 and ψ_3 give us finally

$$x = \sqrt{a^2-c^2} \cdot \frac{dn\,\theta_2\, sn\,\theta_3}{sn\,\theta_1},$$

$$y = \sqrt{a^2-c^2}\, \frac{dn\,\theta_1\, cn\,\theta_2\, cn\,\theta_3}{sn\,\theta_1},$$

$$z = \sqrt{a^2-c^2}\, \frac{cn\,\theta_1\, sn\,\theta_2\, dn\,\theta_3}{sn\,\theta_1}.$$

In the above the modulus k is $= \sqrt{\frac{a^2-b^2}{a^2-c^2}}$.

We have also for the quantities A_1, B_1, C_1 the values

$$A_1 = \frac{2\,abc}{(a^2-c^2)^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} \mathrm{sn}^2\,\theta_1\,d\theta_1,$$

$$B_1 = \frac{2\,abc}{(a^2-c^2)^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} \left(\frac{1-\mathrm{dn}^2\,\theta_1}{\mathrm{dn}^2\,\theta_1}\right) d\theta_1,$$

$$C_1 = \frac{2\,abc}{(a^2-c^2)^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{sn}^2\,\theta_1}{\mathrm{cn}^2\,\theta_1}\,d\theta_1;$$

or

$$A_1 = -\frac{I}{k^2}\left\{\frac{k\theta_1}{K}\frac{dE}{dk} + \frac{\Theta'(\theta_1)}{\Theta(\theta_1)}\right\},$$

$$B_2 = \frac{I}{k^2}\left\{\frac{\theta_1}{k}\frac{d\log K}{dk} + \frac{1}{k'^2}\frac{\Theta'(\theta_1+K)}{\Theta(\theta_1+K)}\right\},$$

$$C_2 = \frac{I}{k'^2}\left\{\frac{d}{d\theta_1}\log H(\theta_1+K) - \frac{E}{K}\,\theta_1\right\},$$

where for brevity I have written

$$I = \frac{2abc}{(a^2-c^2)^{\frac{3}{2}}}.$$

The above transformations are given in full in an article, "On the Motion of an Ellipsoid in a Fluid," American Journal of Mathematics, Vol. II. We find, by the same transformations,

$$\Psi = 2\,\pi abc\int_\sigma^\infty \frac{d\psi_1}{N_1} = \frac{4\,\pi abc}{\sqrt{a^2-c^2}}\,\theta_1,$$

and also

$$\Psi_0 = 2\,\pi abc\int_0^\infty \frac{d\psi_1}{N_1} = \frac{4\,\pi abc}{\sqrt{a^2-c^2}}\,K,$$

K denoting the complete elliptic integral of the first kind. Equating now to zero the found values of u, v, w, we have at once, by means of the foregoing transformations,

$$\alpha = \frac{(a^2-c^2)^{\frac{3}{2}}}{2\,abc}\cdot\frac{\lambda}{a^2\int_0^{\frac{\pi}{2}} \mathrm{sn}^2\,\theta_1 d\theta_1 - 2\,\pi(a^2-c^2)\,K}$$

and

$$\beta = a^2\alpha.$$

The computation of the force necessary to keep the body in place, or the force which must be applied to overcome the resistance of the fluid if the body is moving through it, could be in this case, as in the former, computed by means of the dissipation-function; but the process would be rather tedious and complicated. We can, however, without much difficulty, compute the pressure over the surface of the body supposed in motion. Observe that since Ψ is a surface potential corresponding to a surface density